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SUPER-BURNETT CORRECTIONS TO THE STRESS TENSOR AND THE HEAT FLUX IN A GAS OF MAXWELLIAN MOLECULES†

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Expressions are obtained for the super-Burnett corrections to the stress tensor and the heat flux in a gas consisting of Maxwellian molecules in two special cases: one-dimensional flows and spatial, but weakly perturbed flows.

THE CHAPMAN–ENSKOG method for solving the Boltzmann equation leads to a sequence of hydrodynamic equations [1]. The Euler equations are obtained in the zeroth approximation (with respect to the Knudsen number), the Navier–Stokes equations in the first approximation and the Burnett equations in the second approximation. It is customary to refer to the next approximation as the super-Burnett approximation. The contribution of each approximation to the hydrodynamic equations reduces to the corresponding additions to the stress tensor and heat flux

$$p_{\alpha\beta}=p\delta_{\alpha\beta}+p_{\alpha\beta}^{(1)}+p_{\alpha\beta}^{(2)}+p_{\alpha\beta}^{(3)}+\ldots$$
$$\mathbf{q}=0+\mathbf{q}^{(1)}+\mathbf{q}^{(2)}+\mathbf{q}^{(3)}+\ldots$$

Here p is the pressure $p_{\alpha\beta}^{(1)}$, and $\mathbf{q}^{(1)}$ are the Navier–Stokes and Fourier relationships [1] and $p_{\alpha\beta}^{(2)}$, $\mathbf{q}^{(2)}$ and $p_{\alpha\beta}^{(3)}$, $\mathbf{q}^{(3)}$ are the Burnett [1] and super-Burnett [2] corrections to the stress tensor and heat flux.

In recent years, the equations of the Burnett and super-Burnett approximations have been used to solve a number of gas-dynamic problems. In particular, the problem of the propagation of sonic vibrations in simple gases [3] and mixtures of gases [4]‡ and the problem of the structure of a shock wave [5, 6]§ have been solved in one-dimensional formulation. The conclusion was drawn [6] that solutions of the Burnett equations were closer to experimental data and to the results obtained by the method of direct statistical simulation using the Monte-Carlo method than the solutions of the Navier–Stokes equations while there was a deterioration in the agreement when the super-Burnett equations were used. However, the correctness of the latter conclusion in [6] is doubtful since the expressions for the super-Burnett corrections to the stress tensor and heat flux used contain a number of errors.

Expressions for the super-Burnett corrections to the stress tensor and heat flux are derived in [2] for the case of Maxwellian molecules. However, the calculations were not completely finished since, in a number of terms, the dependence on the gradients of the hydrodynamic quantities is expressed implicitly in terms of the operators D_0/Dt and $\partial_1/\partial t$ (see (1) and (2)). In this paper, these calculations are

†*Prikl. Mat. Mekh.* Vol. 57, No. 3, pp. 168–171, 1993.

‡See also: SHAVALIYEV M. Sh., Transport phenomena in gaseous mixtures in the Burnett and super-Burnett approximations. Candidate dissertation, Novosibirsk, 1978,

§See also: SIMON C. E., Theory of shock structure in a Maxwell gas based on the Chapman–Enskog development through Super-Super-Burnett order. Ph.D. thesis, University of Colorado, CO, 1976.

completed in two special cases. One-dimensional expressions and spatial expressions linearized with respect to the gradients are obtained in explicit form for the super-Burnett corrections to the stress tensor and heat flux.

The expressions for the super-Burnett corrections to the stress tensor and heat flux have the form [2]

$$p_{\alpha\beta}^{(3)} = -\frac{\mu}{p}\left(\frac{\partial_1}{\partial t} p_{\alpha\beta}^{(1)} + \frac{D_0}{Dt} p_{\alpha\beta}^{(2)} + p_{\alpha\beta}^{(2)} \operatorname{div} \mathbf{u} + 2\left\langle p_{\alpha\gamma}^{(2)} \frac{\partial u_\beta}{\partial r_\gamma}\right\rangle + \frac{\partial}{\partial r_\gamma} P_{\alpha\beta\gamma}^{(2)} + \frac{4}{5}\left\langle \frac{\partial q_\alpha^{(2)}}{\partial r_\beta}\right\rangle\right) \tag{1}$$

$$q_\alpha^{(3)} = -\frac{3\mu}{2p}\left[\frac{\partial_1}{\partial t} q_\alpha^{(1)} + \frac{D_0}{Dt} q_\alpha^{(2)} + \frac{10}{3} q_\alpha^{(2)} \operatorname{div} \mathbf{u} + q_\beta^{(2)} \frac{\partial u_\alpha}{\partial r_\beta} + \frac{4}{5} e_{\alpha\beta} q_\beta^{(2)} + \frac{5p}{2\rho}\frac{\partial}{\partial r_\beta} p_{\beta\alpha}^{(2)} - \frac{1}{\rho} p_{\alpha\beta}^{(1)} \frac{\partial}{\partial r_\gamma} p_{\gamma\beta}^{(1)} - \frac{1}{\rho} p_{\alpha\beta}^{(2)} \frac{\partial p}{\partial r_\beta} + P_{\alpha\beta\gamma}^{(2)} e_{\gamma\beta} + \frac{1}{2}\frac{\partial}{\partial r_\beta}\left(P_{21\alpha\beta}^{(2)} + \frac{1}{3} P_{41}^{(2)} \delta_{\alpha\beta}\right)\right] \tag{2}$$

$$R = k/m, \quad p = \rho RT, \quad e_{\alpha\beta} = \left\langle \partial u_\alpha / \partial r_\beta \right\rangle$$

$$P_{\alpha\beta\gamma}^{(2)} = \frac{4\mu^2}{\rho}\left[\left\langle \frac{\partial}{\partial r_\alpha} e_{\beta\gamma}\right\rangle + \left(\frac{5}{2} + \frac{T}{\mu}\frac{d\mu}{dT}\right)\frac{1}{T}\left\langle e_{\alpha\beta}\frac{\partial T}{\partial r_\gamma}\right\rangle - \frac{1}{p}\left\langle e_{\alpha\beta}\frac{\partial p}{\partial r_\gamma}\right\rangle\right], \quad P_{41}^{(2)} = \frac{p\mu^2}{\rho^2}\left[20\frac{\rho}{p} e_{\alpha\beta} e_{\beta\alpha} + \frac{45}{T}\nabla^2 T - \frac{4}{5pT}\frac{\partial T}{\partial r_a}\frac{\partial p}{\partial r_a} + \frac{45}{T^2}\left(\frac{7}{2} + \frac{T}{\mu}\frac{d\mu}{dT}\right)(\nabla T)^2\right]$$

Here ρ, T and u are the density, temperature and macroscopic velocity of the gas, μ is the coefficient of viscosity, k is Boltzmann's constant, and $P_{4l\alpha\beta}^{(2)}$ is obtained by multiplying $P_{\alpha\beta}^{(2)}$ [1] by p/ρ and replacing the coefficients $\overline{\omega}_i$ by ε_i, where [2]

$$\varepsilon_1 = \frac{28}{3}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right), \quad \varepsilon_2 = 14, \quad \varepsilon_3 = 39$$

$$\varepsilon_4 = -18, \quad \varepsilon_5 = 39\left(\frac{12}{13} + \frac{T}{\mu}\frac{d\mu}{dT}\right), \quad \varepsilon_6 = \frac{472}{7}$$

The angular brackets denote that the corresponding tensor is completely symmetrized and its trace with respect to any pair of indices is equal to zero. In particular,

$$\left\langle T_{\alpha\beta}\right\rangle \equiv \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) - \frac{1}{3}\delta_{\alpha\beta} T_{\gamma\gamma}$$

$$\left\langle T_{\alpha\beta\gamma}\right\rangle \equiv \frac{1}{6}(T_{\alpha\beta\gamma} + T_{\alpha\gamma\beta} + T_{\beta\alpha\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} + T_{\gamma\beta\alpha}) - \frac{1}{15}[(T_{\alpha\delta\delta} + T_{\delta\alpha\delta} + T_{\delta\delta\alpha})\delta_{\beta\gamma} + (T_{\beta\delta\delta} + T_{\delta\beta\delta} + T_{\delta\delta\beta})\delta_{\alpha\gamma} + (T_{\gamma\delta\delta} + T_{\delta\gamma\delta} + T_{\delta\delta\gamma})\delta_{\alpha\beta}]$$

Summation is carried out everywhere over repeated vector and tensor indices.

In order to obtain explicit expressions for $P_{\alpha\beta}^{(3)}$ and $q_\alpha^{(3)}$ it is necessary to calculate the terms with the operators D_0/Dt and $\partial_1/\partial t$ in (1) and (2). The operators D_0/Dt and $\partial_1/\partial t$ are defined in [1]. The complexity of expressions (1) and (2) makes it difficult to use the hydrodynamic equations of the super-Burnett approximation when solving actual problems. Special cases are therefore considered below when these expressions can be simplified considerably.

In the case of weakly perturbed gas flows, it is possible to carry out a linearization in (1) and (2) with respect to the gradients of the gas dynamic quantities. Omitting the simple, but lengthy calculations, we shall merely present the final results

$$
\begin{aligned}
p_{\alpha\beta}^{(3)} &= \frac{\mu^3}{p\rho}\left(\frac{5}{3}\left\langle \frac{\partial^2 \operatorname{div} \mathbf{u}}{\partial r_\alpha \partial r_\beta}\right\rangle - \frac{4}{3}\nabla^2 e_{\alpha\beta}\right) \\
q_\alpha^{(3)} &= -\frac{\mu^3}{\rho^2}\left[\frac{157}{16T}\nabla^2\left(\frac{\partial T}{\partial r_\alpha}\right) + \frac{5}{8\rho}\nabla^2\left(\frac{\partial \rho}{\partial r_\alpha}\right)\right]
\end{aligned}
\qquad (3)
$$

In the case of one-dimensional flows, the following expressions are obtained from (1) and (2) for the super-Burnett corrections to the stress tensor and heat flux (a prime denotes a partial derivative with respect to x)

$$
\begin{aligned}
p_{xx}^{(3)} = \frac{\mu^3}{p^2}\Big(&\frac{47}{3}\frac{R}{\rho}T'\rho'u' - \frac{40}{3}\frac{RT}{\rho^2}\rho'^2 u' + \frac{32}{3}\frac{RT}{\rho}\rho'' u' - \frac{2}{3}\frac{RT}{\rho}\rho' u'' - 7\frac{R}{T}T'^2 u' - \\
&-\frac{47}{9}RT'u'' - \frac{31}{9}RT''u' + \frac{2}{9}RTu''' + \frac{16}{27}u'^3\Big)
\end{aligned}
\qquad (4)
$$

$$
\begin{aligned}
q_x^{(3)} = \frac{\mu^3}{p\rho}\Big(&-\frac{9005}{168}\frac{1}{T}T'u'^2 + \frac{271}{21}\frac{1}{\rho}\rho' u'^2 + \frac{421}{42}u'u'' + \frac{917}{8}\frac{R}{\rho T}\rho' T'^2 - \frac{1137}{16}\frac{R}{\rho^2}T'\rho'^2 + \\
&+\frac{397}{16}\frac{R}{\rho}\rho' T'' + \frac{701}{16}\frac{R}{\rho}T'\rho'' - \frac{813}{16}\frac{R}{T^2}T'^3 - \frac{1451}{16}\frac{R}{T}TT'' - \\
&-\frac{157}{16}RT''' - \frac{41}{8}\frac{RT}{\rho^2}\rho'\rho'' - \frac{5}{8}\frac{RT}{\rho}\rho''' + \frac{23}{4}\frac{RT}{\rho^3}\rho'^3\Big)
\end{aligned}
\qquad (5)
$$

For reference purposes, we also present expressions for the Burnett corrections to the stress tensor and heat flux

$$
p_{xx}^{(2)} = \frac{\mu^2}{p}\left(\frac{8}{9}u'^2 - \frac{4}{3}\frac{RT}{\rho} \div \frac{4}{3}\frac{RT}{\rho^2}\rho'^2 - \frac{4}{3}\frac{R}{\rho}\rho' T' + 2\frac{R}{T}T'^2 + \frac{2}{3}RT''\right)
$$

$$
q_x^{(2)} = \frac{\mu^2}{\rho}\left(\frac{95}{8}u'T' - \frac{7}{4}u'' - \frac{2}{\rho}u'\rho'\right).
$$

Comparison of (4) and (5) with the similar expressions in [6] shows that the numerical coefficients of the second and third terms in (4), and the first, second and third terms in (5), differ in magnitude (but not in sign) from those given in [6] and, in fact, these are 64/9, 40/9, 8035/336, 166/21 and 949/168, respectively. Furthermore, the suggestion made in [6] that the numerical coefficient of $\rho T'u'$ in $p_{xx}^{(3)}$ given in Simon's dissertation (see the previous footnote) contains an algebraic error is not confirmed. There is also an error in the expression for the Burnett correction to the stress tensor [6]: the coefficient of u'^2 is taken as being equal to 40/27. This error arose due to the fact that, when calculating the contribution of $\omega_6\langle e_{x\gamma}e_{\gamma x}\rangle = \omega_6(e_{x\gamma}e_{\gamma x} - e_{\beta\gamma}e_{\gamma\beta}/3)$ (see [1]) in formula (14) of [6], where $e_{\beta\gamma} = 0$, $\beta \neq \gamma$, $e_{xx} = \frac{2}{3}u'$ and $e_{yy} = e_{zz} = -\frac{1}{3}u'$, e_{yy} and e_{zz} were erroneously taken as being equal to zero.

In concluding we note that it is necessary to return to the solution of the problem of the structure of the shock wave using the expressions obtained above for the Burnett and Super-Burnett corrections to the stress tensor and heat flux. The simplicity and correctness of the mathematical formulation of this problem and the possibility of comparing the solution with experiments and the results of calculations at the kinetic level enables one to answer the question regarding the existence of the equations of the Burnett and super-Burnett approximations themselves.

Remark. The numbers A_2 and A_4 occur in the Burnett correction to the distribution function [2]. The values of A_1 and A_2 are given in [1], and A_3 and A_4 have been calculated: $A_3 = 0.5862$ and $A_4 = 0.5971$.

REFERENCES

1. CHAPMAN T. and COWLING T., *Mathematical Theory of Inhomogeneous Gases.* IIL, Moscow, 1960.
2. SHAVALIYEV M. Sh., The Burnett approximation to the distribution function and super-Burnett contributions to the stress tensor and heat flux. *Prikl. Mat. Mekh.* **42**, 4, 656–660, 1978.
3. WANG CHANG C. S. and UHLENBECK C. E., On the propagation of sound in monatomic gases. *Studies in Statistical Mechanics* (Edited by J. de Boer and G. E. Uhlenbeck), pp. 43–75. North-Holland, Amsterdam, 1970.
4. FOCH J. D., UHLENBECK G. E. and LOSA M. F., Theory of sound propagation in mixtures of monatomic gases. *Phys. Fluids* **15**, 7, 1224–1232, 1972.
5. FOCH J. D., On higher order hydrodynamic theories of shock structure. *The Boltzmann Equations* (Edited by E. G. Cohen and W. Thirring), pp. 123–140. Springer, Vienna, 1973.
6. FISCO K. A. and CHAPMAN D. R., Comparison of Burnett, Super-Burnett, and Monte Carlo solutions for hypersonic shock structure. *Rarefied Gas Dynamics: Proc. 16th International Symposium* (Edited by E. P. Muntz), pp. 374–395. AIAA, Washington, 1989.

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